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Adiabatic transverse modes in a uniformly rotating fluid

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The two-dimensional adiabatic transverse normal modes of an inviscid compressible fluid, having solid body rotation about the axis of its cylindrical container, are considered. Relative to the rotating fluid there are two trains of harmonic waves, propagating in opposite directions. The first five modes of the first-order harmonic wave and the first mode of the harmonic waves of order two, five, ten, fifteen and twenty have been considered. The period and amplitude of the waves is considerably modified by the rotation. Relative to a fixed coordinate system the angular velocity of the waves initially propagating in the same direction as the rotating fluid, is larger than that of the waves propagating in the opposite direction, as might be expected. However, in the case of the first mode, relative to the rotating fluid, the waves propagating in the opposite direction are the faster.

1. Introduction

The dynamics of rotating fluids, because of their connexion with atmospheric and geophysical phenomena, have been studied by many authors. An account of the effect of rotation on the stability of certain hydrodynamic and magnetohydrodynamic flows has been given by Chandrasekhar (1961). More recently Lighthill (1966) presented a survey of the dynamics of rotating fluids.

In most investigations the effect of fluid compressibility is neglected; and this, in general, simplifies the mathematical problem considerably. However, in some practical applications, such as vortex generators and combustion chambers, we deal with compressible rotating fluids (Swithenbank & Sotter, 1964) and thus it would be of interest to investigate the effect of compressibility on the small oscillations of such flows. Indeed, the present author (Sozou) investigated the symmetrical normal modes of an inviscid perfect fluid, rotating about the axis of its cylindrical container, and showed that rotation increases the frequency of the normal modes and modifies their amplitude considerably. In the present note we consider the transverse (tangential) waves in a perfect inviscid fluid, having solid body rotation about the axis of its cylindrical container.

2. Equations of the problem

We assume that our gas is contained in an infinitely long cylindrical cavity of radius a and rotating with a constant velocity Ω about the axis of symmetry of the cavity.

We use a cylindrical polar co-ordinate system (R, θ, z), fixed in space, with the z-axis along the axis of symmetry of the cavity. We assume that our gas is perfect and the entropy is constant throughout the flow, that is, we assume that the pressure p and density ρ of the gas are connected by the relation

$$p = A\rho^{\gamma},$$

where A is a constant and γ, assumed a constant, is the ratio of the specific heats of the gas. If we non-dimensionalize our quantities by

$$R = ar, \quad C_0(R) = C(r)\,C_0(0), \quad \rho_0(R) = \rho(r)\,\rho_0(0), \quad \Omega = \omega C_0(0)/a, \quad \mathbf{V}_0 = C_0(0)\mathbf{V},$$

where $\mathbf{V}_0$ is the fluid velocity, $C_0(R)$ is the speed of sound and $\rho_0(R)$ is the gas density in the steady state, the integral of the momentum equation becomes

$$C^2 = \rho^{\gamma-1} = 1 + (\gamma-1)\,\omega^2 r^2/2, \tag{1}$$

and $\mathbf{V}$ is given by

$$\mathbf{V} = (0, \omega r, 0). \tag{2}$$

We consider a two-dimensional perturbation ($\partial/\partial z = 0$) of this state. The case of purely radial disturbances ($\partial/\partial\theta = 0$) has been considered elsewhere (Sozou) and here we will be concerned with transverse waves. Thus we let

$$\rho_1 = \rho'(r)\, e^{ik(\omega_2 t - \theta)}$$

and

$$\mathbf{V}_1 = (u, v, 0) = (u'(r), v'(r), 0)\, e^{ik(\omega_2 t - \theta)}.$$

Here the suffix 1 refers to the perturbation quantities. If we now substitute the above relations in the continuity and momentum equations making use of (2), eliminate p by using the equation of state, and omit primes, to a first-order approximation, we obtain the following set of equations:

$$ik(\omega_2 - \omega)\,\rho_1 + \frac{1}{r}\left[\frac{d}{dr}(r\rho u) - ik\rho v\right] = 0, \tag{3}$$

$$ik(\omega_2 - \omega)\,u - 2\omega v + (d/dr)\,(C^2\rho_1/\rho) = 0, \tag{4}$$

$$ik(\omega_2 - \omega)\,v + 2\omega u - ikC^2\rho_1/\rho r = 0. \tag{5}$$

If we eliminate ρ_1 between (3) and (5) and between (4) and (5), and then eliminate v between the resulting pair of equations, we obtain a second-order linear differential equation in u, which by using (1) to eliminate C and ρ becomes

$$r^2(a_0 + a_1 r^2 + a_2 r^4)\,u'' + r(b_0 + b_1 r^2 + b_2 r^4)\,u' + (C_0 + C_1 r^2 + C_2 r^4)\,u = 0, \tag{6}$$

where primes denote differentiation with respect to r and

$$a_0 = 4(\omega_2 - \omega), \quad a_1 = 4(\omega_2 - \omega)\,\{(\gamma-1)\,\omega^2 - (\omega_2 - \omega)^2\},$$

$$a_2 = 4(\omega_2 - \omega)\,(\gamma - 1)\,\omega^2\{(\gamma-1)\,\omega^2 - 2(\omega_2 - \omega)^2\},$$

$$b_0 = 12(\omega_2-\omega), \quad b_1 = 4(\omega_2-\omega)\{(3\gamma-2)\,\omega^2-(\omega_2-\omega)^2\},$$
$$b_2 = (3\gamma-1)\,(\omega_2-\omega)\{(\gamma-1)\,\omega^2-2(\omega_2-\omega)^2\},$$
$$C_0 = 4(\omega_2-\omega)\,(1-k^2),$$
$$C_1 = 4(\omega_2-\omega)\,(1-k^2)\{(\gamma-1)\,\omega^2-2(\omega_2-\omega)^2\}+4\omega_2^2(3\omega_2-5\omega),$$
$$C_2 = \{(\gamma-1)\,\omega^2-2(\omega_2-\omega)^2\}\{(\omega_2-\omega)\,(1-k^2)\,[(\gamma-1)\,\omega^2-2(\omega_2-\omega)^2] + 2\omega_2^2(\omega_2-2\omega)\}.$$

The solution of (6) is subject to the boundary conditions

$$u \text{ finite (for } r \leqslant 1); \quad u(1) = 0. \tag{7}$$

When ω is 0, the solution of (6), which is finite at the origin is $J'_k(k\omega_2 r)$, where J_k is the Bessel function of order k. Thus $k\omega_2$ is given by the zeros of J'_k.

3. Method of computation

We choose γ as 1·4 and for a specified ω we guess an ω_2 and, starting from the origin, we build up a numerical solution of (6) as follows.

Over the first five steps from the origin we use the series expansion form of (6), choosing the coefficient of the lowest power of r to be 1. Thence we build up the solution by using Hamming's step-by-step method (cf. Ralston & Wilf 1961). For a particular k and mode the ω_2 satisfying $u(1) = 0$ is obtained by iteration. The computations of the method described here were performed on the I.C.L. 1907 Computer of Sheffield University.

The coefficient of u'' in (6) vanishes when C^2 is equal to $r^2(\omega_2-\omega)^2$, but u and its derivatives are always finite.

4. Results and discussion

We have taken γ as 1·4 and performed computations for several sets of data. Results of these computations are shown in figures 1–7.

Figure 1 shows the frequency ω_2 of the first normal mode as a function of ω, when k is 1, 2, 5, 10, 15 and 20. Figure 2 shows the frequency ω_2 of the first-order harmonic wave ($k = 1$) for the first five normal modes. It must be noted that in dimensional units the normal mode frequency and the angular velocity of the rotating fluid are $\omega_2 C_0(0)/a$ and $\omega C_0(0)/a$ respectively.

From figures 1 and 2, it is easily seen that for waves initially propagating in the same direction as the rotating fluid (ω and ω_2 positive), ω_2 always increases with ω. In the case of the first mode waves, the increase in ω_2 corresponding to an increase in ω increases with the order of the harmonic, though $\omega_2(\omega)$ is larger, the lower the order of the harmonic is. Our computations also indicate, as can easily be verified by inspection of table 1, that as ω increases all the first mode frequencies tend from above to $\omega + C(1)$.

When k is 1 (figure 2) percentagewise the increase in ω_2, corresponding to an increase in ω, is larger the lower the mode is, in agreement with the results obtained for the symmetrical normal modes (Sozou).

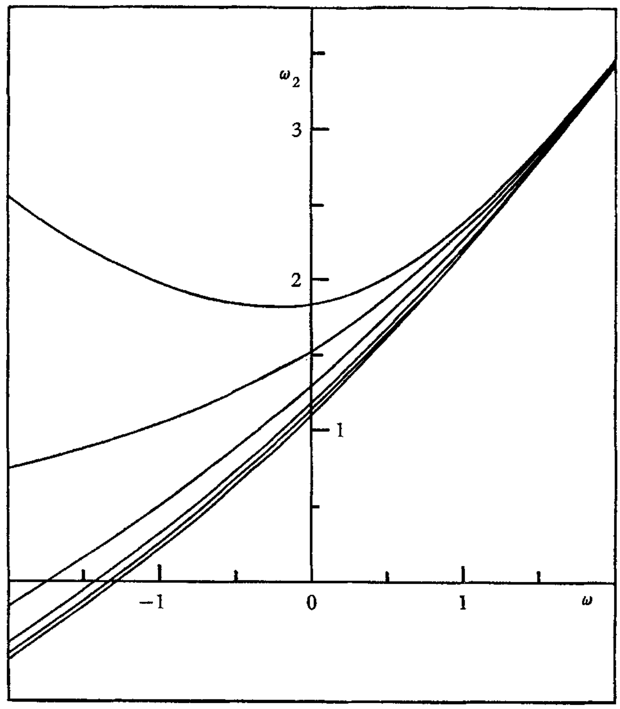

FIGURE 1. First mode frequency ω_2 as a function of ω. The curves, from top to bottom, correspond to k equal to 1, 2, 5, 10, 15 and 20 respectively.

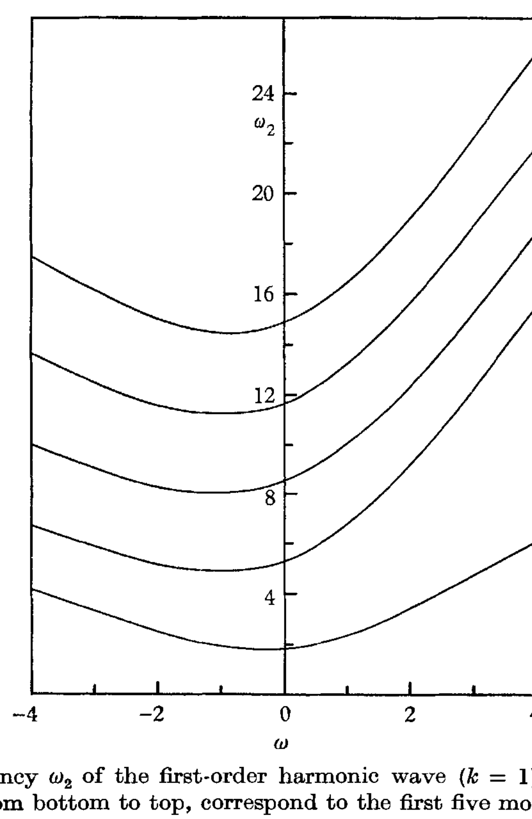

FIGURE 2. Frequency ω_2 of the first-order harmonic wave ($k = 1$) as a function of ω. The curves, from bottom to top, correspond to the first five modes consecutively.

Harmonic (k)	ω	ω_2	$\omega + C(1)$
1	1	2·375	2·095
2	1	2·331	,,
5	1	2·265	,,
10	1	2·219	,,
15	1	2·196	,,
20	1	2·182	,,
1	2	3·460	3·342
2	2	3·459	,,
5	2	3·442	,,
10	2	3·425	,,
15	2	3·415	,,
20	2	3·406	,,
1	3	4·745	4·673
2	3	4·744	,,
1	4	6·1001	6·049
2	4	6·0999	,,
1	6	8·89601	8·864
2	6	8·89596	,,

TABLE 1. First mode frequencies

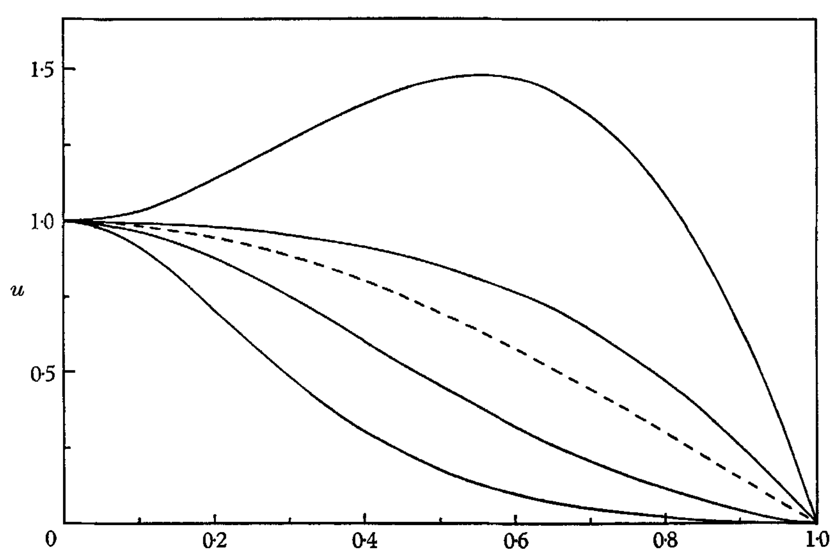

FIGURE 3. First mode amplitude of u (with $u(0) = 1$) as a function of r when k is 1. The curves from top to bottom correspond to ω equal to 4, 2, 0, -2 and -4, respectively.

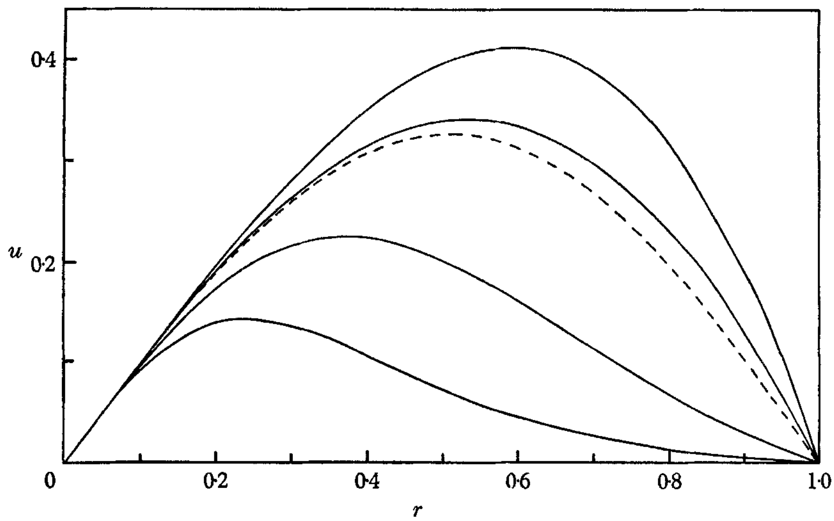

FIGURE 4. First mode amplitude of u (with $u'(0) = 1$) as a function of r when k is 2. The curves from top to bottom correspond to ω equal to 4, 2, 0, -2 and -4, respectively.

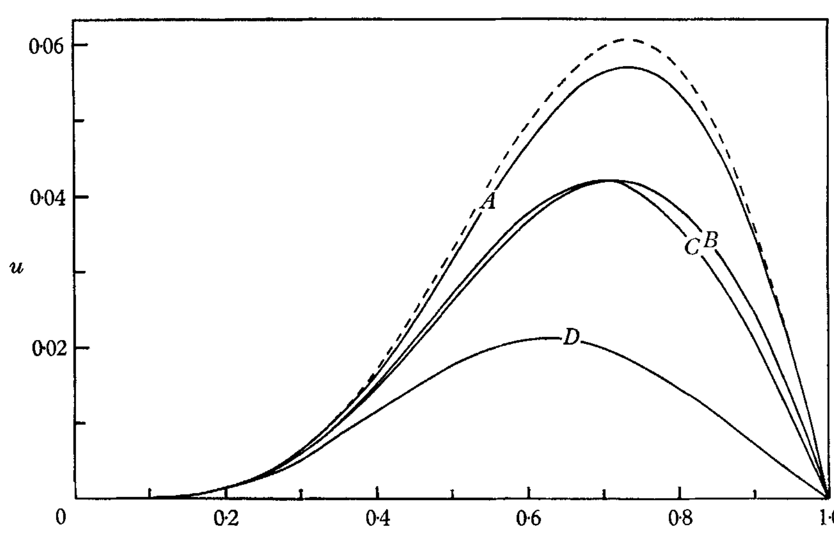

FIGURE 5. First mode amplitude of u (with $u^{\text{iv}}(0) = 4!$) as a function of r when k is 5. The curves A, B, C, D and the broken curve correspond to the cases when ω is 1, 2, -1, -2 and 0, respectively.

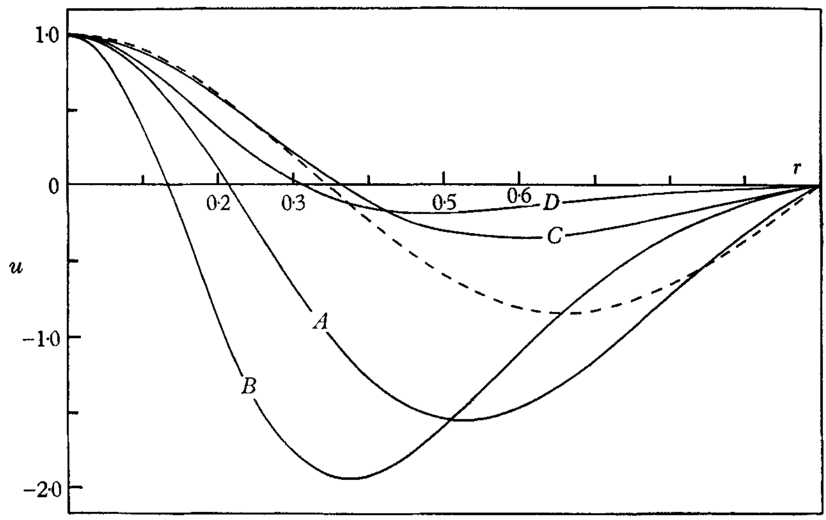

FIGURE 6. Second mode amplitude of u (with $u(0) = 1$) as a function of r when k is 1. The curves A, B, C, D and the broken curve correspond to the cases when ω is 2, 4, -2, -4, and 0, respectively.

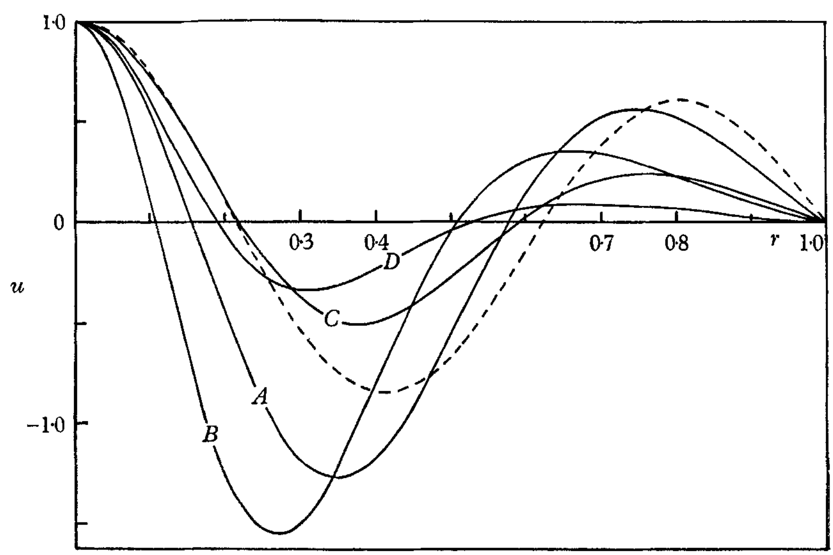

FIGURE 7. Third mode amplitude of u (with $u(0) = 1$) as a function of r when k is 1. The curves A, B, C, D and the broken curve correspond to the cases when ω is 2, 4, -2, -4 and 0, respectively.

For the data considered here, the effect of rotation on the frequency of waves, initially propagating opposite to the direction of rotation (ω negative), is the following.

The frequency of the first-order waves (figure 2) is reduced for slow rotation and reaches a minimum. (For the first mode the minimum occurs when ω is about $-0{\cdot}25$ and for the remaining modes shown in figure 2 the minimum occurs when ω is about -1.) Thence large angular fluid velocities, speed up, not only the waves propagating in the direction of fluid rotation, but all the first-order waves propagating in the opposite direction, as well.

From the first mode waves (figure 1) only the first-order one is eventually speeded up by rotation (this wave tends to rotate as fast as the fluid; when ω is -3, -4 and -6 the corresponding ω_2 is 3·34, 4·22 and 6·12, respectively). The remaining waves are slowed down and for sufficiently large ω and $k > 2$ all the first mode waves are forced to rotate in the same sense as the fluid.

For all first mode waves, $|\omega - \omega_2|$ is smaller when ω is positive, that is, relative to the rotating fluid all the first mode waves, which initially propagate in the same sense as the rotating fluid, are slower than the corresponding waves propagating in the opposite direction. For the other modes this is true only for a range of values of ω.

The effect of rotation on the amplitude of u is shown in figures 3–7. If the magnitude of the disturbance near the origin is specified (for the k-wave we have set $u^{k-1}(0) = (k-1)!$), the amplitude of u, corresponding to a negative ω, is depressed. The amplitude of u, corresponding to a positive ω, increases with ω, until it reaches a maximum, and thence it decreases when ω increases, though, when k is 1 or 2 this amplitude seems to increase continuously with ω.

Positive ω shifts the zeros of u, towards the origin whereas negative ω at first shifts these zeros away from the origin, but as it increases it eventually shifts them towards the origin.

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